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The elementary sources of multipole radiation

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Abstract. Starting from decomposition theorems for arbitrary vector fields, the elementary point sources generating pure multipole radiation are found.

1. Introduction

Multipole radiation has been discussed very many times. An extensive list of references is included in Gray (1978). A discussion of the harmonic case using the usual vector potential is a little awkward but can be achieved, even with considerable elegance (Wallace 1951). Bouwkamp and Casimir (1954) found a straightforward approach based on the differential equations satisfied by the radial components $\boldsymbol{x} \cdot \boldsymbol{E}$ and $\boldsymbol{x} \cdot \boldsymbol{B}$. In the course of their analysis, they proved that the divergence-free electric and magnetic fields in empty space could be represented in terms of Debye potentials, in the form

$$\boldsymbol{x} \times \boldsymbol{\nabla} P + \boldsymbol{\nabla} \times (\boldsymbol{x} \times \boldsymbol{\nabla}) S.$$

Nisbet (1955a) found a potential representation of $\boldsymbol{E}$ and $\boldsymbol{B}$ which was valid within the source. He had previously discussed the history and arbitrariness of various scalar and vector potentials in a wider context (Nisbet 1955b, in which references to the earlier work of Debye, Whittaker, Hertz and Bromwich may be found).

In the present treatment decomposition theorems which are true for all vector fields have been dealt with separately (Rowe 1979). It is hoped that in this paper the real simplicity of the electromagnetic part of the multipole argument will show itself when the decomposition theorems are used consistently from the beginning.

In electrostatics one can give a natural distribution-theory definition of singular spherical harmonics and spherical δ functions so that Poisson's equation

$$\nabla^2 \frac{Y_{lm}(\boldsymbol{x})}{r^{2l+1}} = -4\pi\delta_{lm}(\boldsymbol{x})$$

is satisfied (Rowe 1978). The spherical δ function is the point source which generates a pure multipole field. One can do something similar for the sources of multipole radiation.

A general current $\boldsymbol{j}$ can be decomposed uniquely into three parts determined by three scalar potentials P, Q and R. The scalar potentials themselves can be decomposed into spherical δ functions and their derivatives. This is done in § 2. The spherical δ functions are point sources for certain distribution theory solutions of Helmholtz equations. The relevant formulae, derived in § 3, generalise derivative multipole relations of van der Pol (1936) and Erdélyi (1937) to include the behaviour around the singularity. In § 4, harmonic multipole radiation is discussed in what appears to be its simplest possible form. It is generated by $\nabla \times j$, or the potentials P and R. A δ_{lm} term in $R(P)$ is an elementary point source generating pure 2^l electric (magnetic) multipole radiation. Nonharmonic multipole radiation is included for completeness in § 5.

2. Decomposition of the electric current into elementary components

An electric current j, which may be a conserved time-independent current $j(x)$, a time-dependent current $j(x, t)$, or a Fourier component $j(x, \omega)$, is a vector field which may therefore be decomposed (Rowe 1979):

$$j = x \times \nabla P(x) + \nabla Q(x) + xR(x). \tag{1}$$

The implied origin $\mathbf{0}$ is presumed to be near or within the current distribution. The potentials P, Q and R are unique if they are subjected to

$$\int \mathrm{d}\Omega\, P = \int \mathrm{d}\Omega\, Q = 0, \tag{2}$$

in which the integrals are over the surface of any sphere with centre $\mathbf{0}$. The potentials depend on t or ω if j does.

If the current is conserved, $\nabla . j = 0$, (1) takes the simpler form

$$j = x \times \nabla P + \nabla \times (x \times \nabla) S \tag{3}$$

in terms of unique potentials if

$$\int \mathrm{d}\Omega\, P = \int \mathrm{d}\Omega\, S = 0. \tag{4}$$

Whether j is conserved or not, $\nabla \times j$ is conserved, and

$$\nabla \times j = \nabla \times (x \times \nabla) P - x \times \nabla R. \tag{5}$$

In (5), R may be replaced by

$$R' \equiv R - \frac{1}{4\pi} \int \mathrm{d}\Omega\, R, \tag{6}$$

since the difference contributes nothing, and then P and R' satisfy the conditions corresponding to (4).

In any realistic case j will be spatially bounded; that is, it will vanish outside some sphere with centre $\mathbf{0}$. The same will be true of the scalar potentials. They can then be expanded in a series of spherical δ functions and their derivatives (powers of ∇^2).

To get these expansions explicitly one uses the formal series (Rowe 1978)

$$\exp(-x' . \nabla) = 4\pi \sum_{lm} Y^*_{lm}(x') S_l(r'^2 \nabla^2) Y_{lm}(-\nabla) \tag{7}$$

where $Y_{lm}(x') = r'^l Y_{lm}(\hat{x}')$ is a solid spherical harmonic, $Y_{lm}(-\nabla) = (-1)^l Y_{lm}(\nabla)$ is a polynomial differential operator obtained by replacing Cartesian coordinates with derivatives in the solid spherical harmonics, and the function S_l is defined by

$$S_l(x) = \sum_{K=0}^{\infty} \frac{x^K}{2^K K!(2l + 2K + 1)!!}. \tag{8}$$

There is a connection between S_l and the spherical Bessel function j_l given by

$$S_l(-x^2) = j_l(x)/x^l. \tag{9}$$

Equation (9), which follows from (8), will be needed in the next section.

Using (7) one can develop each of the spatially bounded scalar potentials, as $P(\boldsymbol{x})$ is done in

$$\begin{aligned} P(\boldsymbol{x}) &= \int \mathrm{d}\boldsymbol{x}'\, \delta(\boldsymbol{x}-\boldsymbol{x}')P(\boldsymbol{x}') \\ &= \int \mathrm{d}\boldsymbol{x}' \exp(-\boldsymbol{x}'.\boldsymbol{\nabla})\, \delta(\boldsymbol{x})P(\boldsymbol{x}') \\ &= 4\pi \sum_{lm} \int \mathrm{d}\boldsymbol{x}'\, P(\boldsymbol{x}')\, Y^*_{lm}(\boldsymbol{x}')S_l(r'^2\nabla^2)\, Y_{lm}(-\boldsymbol{\nabla})\, \delta(\boldsymbol{x}) \\ &= 4\pi \sum_{lm} (2l-1)!! \int \mathrm{d}\boldsymbol{x}'\, P(\boldsymbol{x}')\, Y^*_{lm}(\boldsymbol{x}')S_l(r'^2\nabla^2)\, \delta_{lm}(\boldsymbol{x}), \end{aligned} \tag{10}$$

where the spherical δ function is defined by

$$\delta_{lm}(\boldsymbol{x}) = \frac{1}{(2l-1)!!} Y_{lm}(-\boldsymbol{\nabla})\, \delta(\boldsymbol{x}). \tag{11}$$

It is the source in Poisson's equation for a singular spherical harmonic

$$\begin{aligned} \nabla^2 \frac{Y_{lm}(\boldsymbol{x})}{r^{2l+1}} &= \nabla^2\left(\frac{1}{(2l-1)!!} Y_{lm}(-\boldsymbol{\nabla})\frac{1}{r}\right) \\ &= -4\pi\, \delta_{lm}(\boldsymbol{x}). \end{aligned} \tag{12}$$

Equation (12) is a distribution-theory formula.

The expansion (10) is really only valid in the context of analytic test functions, but the moments involved are sufficient to characterise the spatially bounded potential P (Dixon 1967).

From (2) and (4) the potentials P, Q and S have no $l = 0$ components, but in general R will have such components. From the uniqueness of the potentials in (1), together with the expansions of the form (10), we can identify independent elementary currents in terms of which a general current $\boldsymbol{j}$ may be regarded as built up:

(*a*) $(\boldsymbol{x}\times\boldsymbol{\nabla})\nabla^{2K}\,\delta_{lm}(\boldsymbol{x})$ $\qquad (l>0, K\geqslant 0)$

(*b*) $\boldsymbol{\nabla}\nabla^{2K}\,\delta_{lm}(\boldsymbol{x})$ $\qquad (l>0, K\geqslant 0)$

(*c*) $\boldsymbol{x}\nabla^{2K}\,\delta_{lm}(\boldsymbol{x})$ $\qquad (l\geqslant 0, K\geqslant 0$ except $l = K = 0)$.

In (*c*) the case $l = K = 0$ is excluded because $\boldsymbol{x}\,\delta(\boldsymbol{x}) = 0$ (in general, $x^n\,\delta^{(m)}(x) = 0$ if $n > m$). The currents (*a*) are automatically conserved. If $\boldsymbol{j}$ is conserved, it is composed of currents (*a*) and combinations of (*b*) and (*c*) of the form

(*d*) $\boldsymbol{\nabla}\times(\boldsymbol{x}\times\boldsymbol{\nabla})\nabla^{2K}\,\delta_{lm}(\boldsymbol{x})$ $\qquad (l>0, K\geqslant 0)$.

One may consider finite combinations of the elementary currents or infinite combinations corresponding to spatially bounded sources.

The two simplest examples of elementary currents are those which give rise to electric and magnetic dipole radiation.

For an oscillating electric dipole,

$$j(x,\omega) = -i\omega p\,\delta(x) \qquad \mathrm{Re}(e^{-i\omega t} j(x,\omega)) = -\omega \sin \omega t p\,\delta(x)$$

$$\rho(x,\omega) = -p.\nabla\,\delta(x) \qquad \mathrm{Re}(e^{-i\omega t}\rho(x,\omega)) = -\cos\omega t p.\nabla\,\delta(x).$$

Since

$$p.\nabla(x\,\delta(x)) = 0 = p\,\delta(x) + x(p.\nabla)\,\delta(x),$$

the only non-zero potential for this case is

$$R(x) = +i\omega p.\nabla\,\delta(x). \tag{13}$$

For an oscillating magnetic dipole, $\rho = 0$ and

$$j(x,\omega) = -cm\times\nabla\,\delta(x) \qquad \mathrm{Re}(e^{-i\omega t} j) = -c\cos\omega t m\times\nabla\,\delta(x).$$

The only non-zero potential is

$$P(x) = cm.\nabla\,\delta(x). \tag{14}$$

3. Elementary solutions of inhomogeneous Helmholtz equations

When the potential functions of the last section are used, as they will be in the next section, to discuss multipole radiation, Helmholtz's equation,

$$(\nabla^2 + k^2) f(x) = -4\pi g(x) \tag{15}$$

arises. The source, $g(x)$, is a potential function which may be developed in a series of terms $\nabla^{2K}\,\delta_{lm}(x)$ as in equation (10). We want to find the solution of (15) for the case when the source function is a single term of the series.

With the physical applications in mind, only a solution of (15) satisfying the outgoing wave boundary condition is considered. For sufficiently smooth $g(x)$, this solution is

$$f(x) = \int dx' \frac{g(x')\exp(ik|x-x'|)}{|x-x'|}. \tag{16}$$

For $r' \equiv |x'| < r \equiv |x|$, we have the familiar expansion (Jackson 1975, p 742)

$$\frac{\exp(ik|x-x'|)}{|x-x'|} = 4\pi ik \sum_{lm} Y_{lm}(x) \frac{h_l^{(1)}(kr) j_l(kr') Y^*_{lm}(x')}{(rr')^l}. \tag{17}$$

If $r > |x'|$ where $g(x') \neq 0$ we can combine (16) and (17) to give a multipole expansion

$$f(x) = 4\pi ik \sum_{lm} \frac{Y_{lm}(x) h_l^{(1)}(kr)}{r^l} \int dx'\, g(x') Y^*_{lm}(\hat{x}') j_l(kr'). \tag{18}$$

On the other hand, the RHS of (17) may be regarded as a Taylor series (in x') for the LHS, more specifically as a 'spherical Taylor series' if we compare it with the general development of an analytic function $\Psi(x')$ obtained by using (7):

$$\begin{aligned}\Psi(x') &= \exp(+x'.\nabla')\Psi(\mathbf{0}) \\ &= 4\pi \sum_{lm} Y^*_{lm}(x') S_l(r'^2\nabla'^2) Y_{lm}(\nabla')\Psi(\mathbf{0}).\end{aligned} \tag{19}$$

It is understood in (19) that in the expansion of $S_l(r'^2\nabla'^2)$, $(r'^2\nabla'^2)^K$ is interpreted as $r'^{2K}(\nabla'^2)^K$.

Comparing (17) and (19)

$$\mathrm{i}kY_{lm}(\boldsymbol{x})\frac{h_l^{(1)}(kr)j_l(kr')}{(rr')^l}=S_l(r'^2\nabla'^2)\,Y_{lm}(\boldsymbol{\nabla}')\left.\frac{\exp(\mathrm{i}k|\boldsymbol{x}-\boldsymbol{x}'|)}{|\boldsymbol{x}-\boldsymbol{x}'|}\right|_{\boldsymbol{x}'=0}$$
$$=S_l(r'^2\nabla^2)\,Y_{lm}(-\boldsymbol{\nabla})\frac{\mathrm{e}^{\mathrm{i}kr}}{r}$$
$$=S_l(-r'^2k^2)\,Y_{lm}(-\boldsymbol{\nabla})\frac{\mathrm{e}^{\mathrm{i}kr}}{r}$$
$$=\frac{j_l(kr')}{(kr')^l}\,Y_{lm}(-\boldsymbol{\nabla})\frac{\mathrm{e}^{\mathrm{i}kr}}{r}. \tag{20}$$

In the last two steps, $(\nabla^2+k^2)(\mathrm{e}^{\mathrm{i}kr}/r)=0\ (r>0)$ and equation (9) have been used. From (20) we have

$$Y_{lm}(-\boldsymbol{\nabla})\frac{\mathrm{e}^{\mathrm{i}kr}}{r}=\mathrm{i}k^{l+1}\frac{Y_{lm}(\boldsymbol{x})h_l^{(1)}(kr)}{r^l} \tag{21}$$

for $r>0$. Using the small argument form for $h_l^{(1)}$, and letting $k\to 0$ for fixed r, (21) reduces to

$$Y_{lm}(-\boldsymbol{\nabla})\frac{1}{r}=\frac{(2l-1)!!\,Y_{lm}(\boldsymbol{x})}{r^{2l+1}}\qquad (r>0),$$

a distribution-theory form of which has been quoted in (12).

Equation (21), derived for $r>0$, may be compared with the somewhat different forms given by van der Pol (1936) and Erdélyi (1937). The RHS may be expressed in several different ways as a differential operator acting on $\mathrm{e}^{\mathrm{i}kr}/r$ because this function satisfies the homogeneous Helmholtz equation for $r\neq 0$. The differences become important if we want to generalise the formulae by interpreting the derivatives in the distribution theory sense.

We adopt (21) as a distribution theory definition of the RHS, the derivatives on the left being interpreted as those of distribution theory

$$\left(Y_{lm}(-\boldsymbol{\nabla})\frac{\mathrm{e}^{\mathrm{i}kr}}{r},\phi\right)\equiv\left(\frac{\mathrm{e}^{\mathrm{i}kr}}{r},\,Y_{lm}(\boldsymbol{\nabla})\phi\right)$$
$$=\int \mathrm{d}\boldsymbol{x}\,\frac{\mathrm{e}^{\mathrm{i}kr}}{r}\,Y_{lm}(\boldsymbol{\nabla})\phi(\boldsymbol{x}). \tag{22}$$

If the test function ϕ is zero in a neighbourhood of the origin, we may write

$$\left(Y_{lm}(-\boldsymbol{\nabla})\frac{\mathrm{e}^{\mathrm{i}kr}}{r},\phi\right)=\int \mathrm{d}\boldsymbol{x}\left(Y_{lm}(-\boldsymbol{\nabla})\frac{\mathrm{e}^{\mathrm{i}kr}}{r}\right)\phi(\boldsymbol{x}) \tag{23}$$

and recover the function form of (21) in the region $r>0$. Again, for a test function that vanishes near $r=0$, we have

$$\left(S_l(r'^2\nabla^2)\,Y_{lm}(-\boldsymbol{\nabla})\frac{\mathrm{e}^{\mathrm{i}kr}}{r},\phi\right)=\int \mathrm{d}x\,S_l(-k^2r'^2)\,Y_{lm}(-\boldsymbol{\nabla})\frac{\mathrm{e}^{\mathrm{i}kr}}{r}\phi$$
$$=\frac{j_l(kr')}{(kr')^l}\left(Y_{lm}(-\boldsymbol{\nabla})\frac{\mathrm{e}^{\mathrm{i}kr}}{r},\phi\right). \tag{24}$$

With the distribution-theory interpretation of (21) we can deduce from (11) and

$$(\nabla^2 + k^2)\frac{e^{ikr}}{r} = -4\pi\,\delta(\boldsymbol{x}) \tag{25}$$

that

$$(\nabla^2 + k^2)\frac{Y_{lm}(-\nabla)}{(2l-1)!!}\frac{e^{ikr}}{r}$$
$$= (\nabla^2 + k^2)\frac{ik^{l+1}}{(2l-1)!!}\frac{Y_{lm}(\boldsymbol{x})h_l^{(1)}(kr)}{r^l} = -4\pi\,\delta_{lm}(\boldsymbol{x}) \tag{26}$$

and

$$(\nabla^2 + k^2)\frac{ik^{l+1}}{(2l-1)!!}\nabla^{2K}\frac{Y_{lm}(\boldsymbol{x})h_l^{(1)}(kr)}{r^l} = -4\pi\nabla^{2K}\,\delta_{lm}(\boldsymbol{x}). \tag{27}$$

Equations (26) and (27) relate elementary sources to their causal, outgoing responses.

If $g(\boldsymbol{x})$ in (15) is expanded as in (10), so that the former equation reads

$$(\nabla^2 + k^2)f(\boldsymbol{x}) = 4\pi\sum_{lm}(2l-1)!!\int \mathrm{d}\boldsymbol{x}'\, g(\boldsymbol{x}')Y^*_{lm}(\boldsymbol{x}')S_l(r'^2\nabla^2)(-4\pi\,\delta_{lm}(\boldsymbol{x})), \tag{28}$$

we may formally solve it by using (26):

$$f(\boldsymbol{x}) = \sum_{lm} 4\pi\int \mathrm{d}\boldsymbol{x}'\, g(\boldsymbol{x}')Y^*_{lm}(\boldsymbol{x}')S_l(r'^2\nabla^2)ik^{l+1}\frac{Y_{lm}(\boldsymbol{x})h_l^{(1)}(kr)}{r^l}. \tag{29}$$

For $\boldsymbol{x}$ outside the region where $g(\boldsymbol{x}') \neq 0$, (24) converts (29) back to (18).

4. Harmonic multipole radiation

Maxwell's equations for a Fourier component with time dependence $e^{-i\omega t}$ are (Jackson 1975)

$$\begin{aligned} &\nabla\cdot\boldsymbol{E} = 4\pi\rho && \nabla\cdot\boldsymbol{B} = 0 \\ &\nabla\times\boldsymbol{E} = i(\omega/c)\boldsymbol{B} && \nabla\times\boldsymbol{B} = -i(\omega/c)\boldsymbol{E} + (4\pi/c)\boldsymbol{j} \end{aligned} \tag{30}$$

in Gaussian units. Current conservation implies

$$\nabla\cdot\boldsymbol{j} - i\omega\rho = 0. \tag{31}$$

For $\omega \neq 0$ we therefore have

$$\nabla\cdot[\boldsymbol{E} + i(4\pi/\omega)\boldsymbol{j}] = \nabla\cdot\boldsymbol{B} = 0 \tag{32}$$

and

$$\nabla\times[\boldsymbol{E} + i(4\pi/\omega)\boldsymbol{j}] = i(\omega/c)\boldsymbol{B} + i(4\pi/\omega)\nabla\times\boldsymbol{j} \tag{33}$$

$$\nabla\times\boldsymbol{B} = -i(\omega/c)[\boldsymbol{E} + i(4\pi/\omega)\boldsymbol{j}]. \tag{34}$$

The two divergence-free fields in (32) may be decomposed by uniquely definable scalar potentials as in (3), (4):

$$\boldsymbol{E} + i(4\pi/\omega)\boldsymbol{j} = \boldsymbol{x}\times\nabla W + \nabla\times(\boldsymbol{x}\times\nabla)X \tag{35}$$

$$\boldsymbol{B} = \boldsymbol{x} \times \boldsymbol{\nabla} Y + \boldsymbol{\nabla} \times (\boldsymbol{x} \times \boldsymbol{\nabla}) Z \tag{36}$$

(compare Bouwkamp and Casimir 1954, Nisbet 1955a, Gray 1978).

Substituting (35) and (36) in (33) and (34), and using the decomposition (5) with (6) for $\boldsymbol{\nabla} \times \boldsymbol{j}$, we get, by identifying the unique potentials,

$$W = \mathrm{i}(\omega/c)Z + \mathrm{i}(4\pi/\omega)P$$

$$-\nabla^2 X = \mathrm{i}(\omega/c)Y - \mathrm{i}(4\pi/\omega)R'$$

$$Y = -\mathrm{i}(\omega/c)X$$

$$-\nabla^2 Z = -\mathrm{i}(\omega/c)W.$$

Using the decomposition (1) for $\boldsymbol{j}$, Maxwell's equations simplify to the expressions

$$\boldsymbol{E} = \mathrm{i}(\omega/c)\boldsymbol{x} \times \boldsymbol{\nabla} Z + \boldsymbol{\nabla} \times (\boldsymbol{x} \times \boldsymbol{\nabla}) X - \mathrm{i}(4\pi/\omega)(\boldsymbol{\nabla} Q + \boldsymbol{x} R) \tag{37}$$

$$\boldsymbol{B} = -\mathrm{i}(\omega/c)\boldsymbol{x} \times \boldsymbol{\nabla} X + \boldsymbol{\nabla} \times (\boldsymbol{x} \times \boldsymbol{\nabla}) Z \tag{38}$$

for the fields in terms of the potentials, and the scalar Helmholtz equations relating the $\boldsymbol{E}$, $\boldsymbol{B}$ potentials to the current potentials

$$[\nabla^2 + (\omega^2/c^2)]X = \mathrm{i}(4\pi/\omega)R' \tag{39}$$

$$[\nabla^2 + (\omega^2/c^2)]Z = -(4\pi/c)P. \tag{40}$$

Outside the source, $\boldsymbol{j}$ vanishes, as do the potentials P, Q, R, so the expressions for $\boldsymbol{E}$ and $\boldsymbol{B}$ take a simple and symmetric form. Electric multipole radiation, satisfying $\boldsymbol{x} . \boldsymbol{B} = 0$, is generated by the current potential R', and magnetic multipole radiation is generated by P. The multipole expansion for $\boldsymbol{E}$ and $\boldsymbol{B}$ arises by writing the solutions of (39) and (40) in the form (18). Elementary solutions, for currents determined by a single term $\nabla^{2K} \delta_{lm}(\boldsymbol{x})$ in R' or P, are obtained by using (27).

For an oscillating electric dipole,

$$\boldsymbol{j}(\boldsymbol{x}, \omega) = -\mathrm{i}\omega \boldsymbol{p}\, \delta(\boldsymbol{x})$$

$$\rho(\boldsymbol{x}, \omega) = -\boldsymbol{p} . \boldsymbol{\nabla}\, \delta(\boldsymbol{x})$$

and

$$R(\boldsymbol{x}) = R'(\boldsymbol{x}) = +\mathrm{i}\omega \boldsymbol{p} . \boldsymbol{\nabla}\, \delta(\boldsymbol{x}), \tag{13}$$

so that, from (39) and (25),

$$X = \boldsymbol{p} . \boldsymbol{\nabla} \frac{\mathrm{e}^{\mathrm{i}kr}}{r} \qquad (k \equiv \omega/c). \tag{41}$$

Outside the source,

$$\boldsymbol{E} = \boldsymbol{\nabla} \times (\boldsymbol{x} \times \boldsymbol{\nabla}) \boldsymbol{p} . \boldsymbol{\nabla} \frac{\mathrm{e}^{\mathrm{i}kr}}{r} = -\boldsymbol{\nabla} \times (\boldsymbol{p} \times \boldsymbol{\nabla}) \frac{\mathrm{e}^{\mathrm{i}kr}}{r} \tag{42}$$

$$\boldsymbol{B} = -\mathrm{i}k\boldsymbol{x} \times \boldsymbol{\nabla} \boldsymbol{p} . \boldsymbol{\nabla} \frac{\mathrm{e}^{\mathrm{i}kr}}{r} = \mathrm{i}k\boldsymbol{p} \times \boldsymbol{\nabla} \frac{\mathrm{e}^{\mathrm{i}kr}}{r}. \tag{43}$$

For the magnetic dipole, equation (14) gives $P = c\boldsymbol{m} . \boldsymbol{\nabla}\, \delta(\boldsymbol{x})$, and so $Z = \boldsymbol{m} . \boldsymbol{\nabla}(\mathrm{e}^{\mathrm{i}kr}/r)$. The fields are obtained from (42) and (43) by $\boldsymbol{p} \to \boldsymbol{m}$, $\boldsymbol{B} \to -\boldsymbol{E}$, $\boldsymbol{E} \to \boldsymbol{B}$.

Static fields correspond to $\delta(\omega)$ terms in the Fourier transforms. For the charge density we write $\delta(\omega)\rho_0(\boldsymbol{x})$, and we similarly introduce $\boldsymbol{j}_0$, $\boldsymbol{E}_0$, $\boldsymbol{B}_0$. Maxwell's equations (30) reduce to

$$\boldsymbol{\nabla}.\boldsymbol{E}_0 = 4\pi\rho_0, \qquad \boldsymbol{\nabla}\times\boldsymbol{E}_0 = 0$$
$$\boldsymbol{\nabla}.\boldsymbol{B}_0 = 0, \qquad \boldsymbol{\nabla}\times\boldsymbol{B}_0 = (4\pi/c)\boldsymbol{j}_0$$
$$\boldsymbol{\nabla}.\boldsymbol{j}_0 = 0.$$

The electrostatic multipole field is developed in the usual way by introducing a scalar potential $\boldsymbol{E}_0 = -\boldsymbol{\nabla}\phi_0$. For the divergence-free steady magnetic field we have, as in (36),

$$\boldsymbol{B}_0 = \boldsymbol{x}\times\boldsymbol{\nabla}Y_0 + \boldsymbol{\nabla}\times(\boldsymbol{x}\times\boldsymbol{\nabla})Z_0, \tag{44}$$

so

$$\boldsymbol{\nabla}\times\boldsymbol{B}_0 = \boldsymbol{\nabla}\times(\boldsymbol{x}\times\boldsymbol{\nabla})Y_0 - \boldsymbol{x}\times\boldsymbol{\nabla}(\nabla^2 Z_0). \tag{45}$$

The conserved current $\boldsymbol{j}_0$ has the development (3):

$$\boldsymbol{j}_0 = \boldsymbol{x}\times\boldsymbol{\nabla}P_0 + \boldsymbol{\nabla}\times(\boldsymbol{x}\times\boldsymbol{\nabla})S_0. \tag{46}$$

Therefore, since the potentials are unique,

$$Y_0 = (4\pi/c)S_0, \qquad \nabla^2 Z_0 = -(4\pi/c)P_0. \tag{47}$$

The equation for Z_0 can be solved and expanded in multipoles $Y_{lm}(\boldsymbol{x})/r^{2l+1}$. Since S_0 vanishes outside the source, so does Y_0; and since P_0 vanishes there too, $\nabla^2 Z_0 = 0$. In this region

$$\boldsymbol{B}_0 = \boldsymbol{\nabla}\times(\boldsymbol{x}\times\boldsymbol{\nabla})Z_0 = -\boldsymbol{\nabla}(1 + r\,\partial/\partial r)Z_0, \tag{48}$$

whose multipole expansion has the same form as that for $\boldsymbol{E}_0$.

5. Nonharmonic multipole radiation

For the case when the time dependence is not harmonic we no longer have the relation (31) for ρ in terms of $\boldsymbol{\nabla}.\boldsymbol{j}$, and so we can no longer introduce potentials for the electric field on the basis of (32). Instead we may express the charge density in the form

$$\rho(\boldsymbol{x}, t) = -\boldsymbol{\nabla}.(\boldsymbol{\nabla}Q + \boldsymbol{x}R), \tag{49}$$

from which we get

$$\boldsymbol{\nabla}.(\boldsymbol{E} + 4\pi\boldsymbol{\nabla}Q + 4\pi\boldsymbol{x}R) = 0. \tag{50}$$

From (49)

$$(\partial\rho/\partial t) + \boldsymbol{\nabla}.(\boldsymbol{\nabla}\dot{Q} + \boldsymbol{x}\dot{R}) = 0, \tag{51}$$

so that charge conservation implies that the current has the form

$$\boldsymbol{j}(\boldsymbol{x}, t) = \boldsymbol{x}\times\boldsymbol{\nabla}P + \boldsymbol{\nabla}\times(\boldsymbol{x}\times\boldsymbol{\nabla})S + \boldsymbol{\nabla}\dot{Q} + \boldsymbol{x}\dot{R}. \tag{52}$$

When Q and R have been fixed, a knowledge of $\boldsymbol{j}$ will permit the calculation of P and S (Rowe 1979). Therefore they may be assumed to be known.

Equation (49) has many solutions, a useful one of which is suggested by writing it in the form

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial Q}{\partial r}\right)+(\boldsymbol{x}\times\boldsymbol{\nabla})^2Q+\frac{\partial}{\partial r}(r^3R)+r^2\rho=0. \tag{53}$$

If we introduce the average charge density on the sphere with radius r

$$\bar{\rho}(r,t)\equiv\int\frac{\mathrm{d}\Omega}{4\pi}\rho(\boldsymbol{x},t), \tag{54}$$

then we can define Q to be the unique solution to

$$(\boldsymbol{x}\times\boldsymbol{\nabla})^2Q+r^2(\rho-\bar{\rho})=0 \tag{55}$$

which satisfies $\bar{Q}=0$. Q vanishes outside a sphere which contains the sources, and vanishes faster than r^2 at $r=0$.

When Q has been chosen to satisfy (55), (49) is satisfied by

$$R(\boldsymbol{x},t)=-\frac{1}{r}\frac{\partial Q}{\partial r}-\frac{1}{r^3}\int_0^r r^2\bar{\rho}\,\mathrm{d}r. \tag{56}$$

The first term

$$R'(\boldsymbol{x},t)=-\frac{1}{r}\frac{\partial Q}{\partial r} \tag{57}$$

averages to zero on each sphere with centre $\mathbf{0}$, and vanishes outside the sources; the second term is a function of r and t alone. If $q\equiv 4\pi\int r^2\bar{\rho}\,\mathrm{d}r$ is the total charge, then outside the sources

$$R(\boldsymbol{x},t)=-\frac{q}{4\pi r^3},\qquad \dot{R}=0 \tag{58}$$

and

$$4\pi\boldsymbol{x}R=\boldsymbol{\nabla}q/r\qquad \boldsymbol{\nabla}\cdot(4\pi\boldsymbol{x}R)=0. \tag{59}$$

Using (50) and $\boldsymbol{\nabla}\cdot\boldsymbol{B}=0$ to justify the introduction of potentials, we have the forms

$$\boldsymbol{E}=\boldsymbol{x}\times\boldsymbol{\nabla}W+\boldsymbol{\nabla}\times(\boldsymbol{x}\times\boldsymbol{\nabla})X-4\pi(\boldsymbol{\nabla}Q+\boldsymbol{x}R) \tag{60}$$

$$\boldsymbol{B}=\boldsymbol{x}\times\boldsymbol{\nabla}Y+\boldsymbol{\nabla}\times(\boldsymbol{x}\times\boldsymbol{\nabla})Z. \tag{61}$$

Because of (52), these equations do not reduce to (35) and (36) in the harmonic case unless the potentials are redefined (which, of course, they can be). The time-dependent Maxwell's equations now imply

$$W=-(\dot{Z}/c)\qquad -\nabla^2X+4\pi R'=-(Y/c)$$

and

$$Y=(\dot{X}/c)+(4\pi/c)S\qquad -\nabla^2Z=(\dot{W}/c)+(4\pi/c)P.$$

Therefore

$$\boldsymbol{E}=-(1/c)\boldsymbol{x}\times\boldsymbol{\nabla}\dot{Z}+\boldsymbol{\nabla}\times(\boldsymbol{x}\times\boldsymbol{\nabla})X-4\pi(\boldsymbol{\nabla}Q+\boldsymbol{x}R) \tag{62}$$

$$\boldsymbol{B}=+(1/c)\boldsymbol{x}\times\boldsymbol{\nabla}\dot{X}+\boldsymbol{\nabla}\times(\boldsymbol{x}\times\boldsymbol{\nabla})Z+(4\pi/c)\boldsymbol{x}\times\boldsymbol{\nabla}S \tag{63}$$

with

$$\nabla^2 Z - (\ddot{Z}/c^2) = -(4\pi/c)P \tag{64}$$

$$\nabla^2 X - (\ddot{X}/c^2) = 4\pi[R' + (\dot{S}/c^2)]. \tag{65}$$

Outside the source (62) and (63) reduce to

$$\boldsymbol{E} = -(1/c)\boldsymbol{x} \times \boldsymbol{\nabla}\dot{Z} + \boldsymbol{\nabla} \times (\boldsymbol{x} \times \boldsymbol{\nabla})X + (q\boldsymbol{x}/r^3) \tag{66}$$

$$\boldsymbol{B} = +(1/c)\boldsymbol{x} \times \boldsymbol{\nabla}\dot{X} + \boldsymbol{\nabla} \times (\boldsymbol{x} \times \boldsymbol{\nabla})Z. \tag{67}$$

The retarded solution to (64) is

$$Z(\boldsymbol{x}, t) = \frac{1}{c}\int \mathrm{d}\boldsymbol{x}' \frac{P(\boldsymbol{x}', t - |\boldsymbol{x} - \boldsymbol{x}'|/c)}{|\boldsymbol{x} - \boldsymbol{x}'|}, \tag{68}$$

and the solution of (65) is similar.

The potentials $Z(\boldsymbol{x}, t)$ and $X(\boldsymbol{x}, t)$ can be expanded in spherical harmonics; for Z we have

$$Z(\boldsymbol{x}, t) = \sum_{lm} Z_{lm}(r, t) Y_{lm}(\hat{\boldsymbol{x}}) \tag{69}$$

where

$$Z_{lm}(r, t) = \frac{1}{c}\int \mathrm{d}\boldsymbol{x}' \int \mathrm{d}\Omega \frac{Y^*_{lm}(\hat{\boldsymbol{x}})P(\boldsymbol{x}', t - |\boldsymbol{x} - \boldsymbol{x}'|/c)}{|\boldsymbol{x} - \boldsymbol{x}'|}.$$

Using the addition theorem for spherical harmonics in order to express the Ω integration with respect to $\hat{\boldsymbol{x}}'$ as pole we get (Campbell *et al* 1977)

$$Z_{lm}(r, t) = \frac{2\pi}{c}\int \mathrm{d}\boldsymbol{x}' \; Y^*_{lm}(\hat{\boldsymbol{x}}') \int_{-1}^{+1} \mathrm{d}\nu \frac{P_l(\nu)P[\boldsymbol{x}', t - c^{-1}(r^2 + r'^2 - 2rr'\nu)^{1/2}]}{(r^2 + r'^2 - 2rr'\nu)^{1/2}}.$$

After introducing the angular decomposition for the current potential $P(\boldsymbol{x}, t)$, Z_{lm} takes the form

$$Z_{lm}(r, t) = \frac{2\pi}{c}\int r'^2 \,\mathrm{d}r' \,\mathrm{d}\nu \frac{P_l(\nu)P_{lm}[r', t - c^{-1}(r^2 + r' - 2rr'\nu)^{1/2}]}{(r^2 + r'^2 - 2rr'\nu)^{1/2}}. \tag{70}$$

Outside the source we may change from the variable ν to η given by

$$(r^2 + r'^2 - 2rr'\nu)^{1/2} = r - \eta r'.$$

This transforms (70) to

$$Z_{lm}(r, t) = \frac{2\pi}{cr}\int r'^2 \,\mathrm{d}r' \int_{-1}^{+1} \mathrm{d}\eta \, P_l[\eta + r'(1 - \eta^2)/2r]P_{lm}(r', t - r/c + \eta r'/c). \tag{71}$$

In this form it is easy to check that Z_{lm} satisfies

$$\frac{1}{r}\frac{\partial^2}{\partial r^2} r Z_{lm} - \frac{l(l+1)}{r^2} Z_{lm} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} Z_{lm} = 0, \tag{72}$$

which, from (64) and (69), it must do beyond the sources.

Additional developments of the time-dependent case have been made by Kemmer (1970) and Campbell *et al* (1977).

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